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# Alternative Approximation for Stresses in Plate Structures

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**An accurate nonlinear approximation for stresses caused by membrane and bending loads in plate structures is presented. Although the approximation requires only a first-order sensitivity analysis, it is more accurate than a simple linear Taylor series expansion of the stresses with respect to the design variables, or their reciprocals, and is exact for statically determinate structures. It requires less computation, both in approximate analysis and the corresponding sensitivity analysis, than other approximations with the same order of accuracy. The use of this alternative approximation for plate stresses in the approximation concepts approach to structural optimization results in very rapid design convergence. Examples of the structural optimization of plate structures subject to stress constraints are presented to show the accuracy of this approximation.**

## Introduction

IN the approximation concepts approach to structural optimization, an approximate analysis problem is created, usually based on a first-order sensitivity analysis of the actual problem, and a new design is generated from an optimization based on this approximate problem. Then an actual analysis is performed on the new design, and a new approximate problem is generated. This process is repeated until the design converges. The rate of design convergence is dependent on the accuracy of the approximations of the design constraints.

A common approach used to generate these approximate problems is to expand the design constraints and, possibly, the objective function, in a linear Taylor series with respect to the design variables or their reciprocals. Although this type of approximation is simple, it is not very accurate for stress constraints because it lumps together the effects of changes in the element forces, due to load redistribution, and changes in the element stress recovery parameters (membrane thickness, bending stiffness, and fiber distance). The stress approximation presented in this work separates these effects and, therefore, is more accurate.

## Background

In Ref. 1 the membrane stresses in plate elements were approximated using Taylor series expansions with respect to the reciprocal of the membrane thickness.

$$\tilde{\sigma}_m = \sigma_{m0} + \sum_{i=1}^{NDV} \frac{\partial \sigma_m}{\partial 1/t_i}\left(\frac{1}{t_i} - \frac{1}{t_{i0}}\right) = \sigma_{m0} + \sum_{i=1}^{NDV} \frac{\partial \sigma_m}{\partial X_i}(X_i - X_{i0}) \quad (1)$$

where NDV is the number of design variables, the tilde denotes an approximate quantity, and the subscript zero denotes the values associated with the actual analysis. The partial derivatives are calculated at the actual analysis design. The reciprocals of the membrane thicknesses can be thought of as intermediate variables. Note that although the approximate stresses are linear functions of the intermediate variables $X$, they are nonlinear functions of the actual design variables. It was pointed out in Ref. 1 that this approximation is exact for statically determinate structures. This is because the stress due to membrane loading, in any direction, in a plate element is just

$$\sigma_m = N/t \quad (2)$$

where $N$ is the membrane force in that direction. However, using the reciprocal of the thickness as the intermediate variable does not generate an exact approximation of bending stresses in statically determinate structures. This is because the bending stress is a function of the reciprocal of the square of the thickness. Note that

$$\sigma_b = (-Mz)/D \quad (3)$$

where

$$D = t^3/12 \quad \text{and} \quad z = \pm t/2 \quad (4)$$

so that

$$\sigma_b = \pm 6M/t^2 \quad (5)$$

The force approximation method for stress constraints, which was introduced in Ref. 2 for beam elements, was applied to plate elements in Ref. 3. In this method the element forces, both membrane and bending, are approximated using a Taylor series expansion with respect to intermediate variables.

$$\tilde{N} = N_0 + \sum_{i=1}^{NIV} \frac{\partial N}{\partial X_i}(X_i - X_{i0})$$
$$\tilde{M} = M_0 + \sum_{i=1}^{NIV} \frac{\partial M}{\partial X_i}(X_i - X_{i0}) \quad (6)$$

where NIV is the number of intermediate variables. The intermediate variables are chosen to be the membrane thickness $t$ and bending stiffness $D$. The approximate forces are used with the exact values of the stress recovery parameters to calculate the approximate stresses.

$$\tilde{\sigma} = \frac{\tilde{N}}{t} - \frac{\tilde{M}z}{D} \quad (7)$$

Note that this approximation is exact for statically determinate structures, even with bending, because the approximate forces are exact. It also captures the cross coupling between the changes in the element forces, due to load redistribution,

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and changes in the element stress recovery parameters. One drawback associated with this approximation is that two forces must be approximated for each stress. This also results in the calculation and storage of twice as many derivatives.

It was shown in Ref. 4 that the stresses in beam elements could be approximated by a Taylor series expanded with respect to intermediate variables $X$, called generalized variables in Ref. 4, consisting of the reciprocals of the element section properties $Z$ and the stress recovery parameters $S$ as well as shape design variables $V$. The stress at any point in a symmetric beam loaded in biaxial bending and tension is

$$\sigma = \frac{-M_1 c_1}{I_1} - \frac{M_2 c_2}{I_2} + \frac{P}{A} \tag{8}$$

The stress recovery parameters $S$ are

$$S_1 = \frac{c_1}{I_1}, \qquad S_2 = \frac{c_2}{I_2}, \qquad S_3 = \frac{1}{A} \tag{9}$$

The reciprocals of the section properties $Z$ are

$$Z_1 = \frac{1}{I_1}, \qquad Z_2 = \frac{1}{I_2}, \qquad Z_3 = \frac{1}{A} \tag{10}$$

The intermediate variables for any beam element are just the combination of both $S$ and $Z$ as well as $V$. Note that the stresses are linear functions of $S$. Also note that the stresses are linear functions of the element forces which are usually close to being linear functions of $Z$. The derivatives of the stress with respect to the stress recovery parameters, at the current design point, are

$$\frac{\partial \sigma}{\partial S_1} = -M_{10} \qquad \frac{\partial \sigma}{\partial S_2} = -M_{20} \qquad \frac{\partial \sigma}{\partial S_3} = P_0 \tag{11}$$

Note that only three derivatives with respect to the $S$ terms exist for each element stress that is being approximated. The partial derivative of the stress with respect to the reciprocal of a section property of any element in the structure is determined from Eq. (8). Noting that the element forces are functions of the reciprocal section properties, the partial derivative is

$$\frac{\partial \sigma}{\partial Z} = \frac{-c_{10}}{I_{10}} \frac{\partial M_1}{\partial Z} - \frac{c_{20}}{I_{20}} \frac{\partial M_2}{\partial Z} + \frac{1}{A_0} \frac{\partial P}{\partial Z} \tag{12}$$

It is seen that $S$ can be thought of as local (element) intermediate variables and that $Z$ and $V$ can be thought of as global (structural) intermediate variables. Note that the derivatives with respect to $S$ are inexpensive to calculate since they do not involve the global equations. Also note that although $S_3$ and $Z_3$ have the same value, $S_3$ is a stress recovery parameter and $Z_3$ is a reciprocal section property. It can be seen that the beam stress approximation formulated using this set of intermediate variables ($S$ and $Z$) is exact for statically determinate structures.

## Approximate Stress Formulation

The development of the approximate plate stress formulation will now be presented in detail. The state of stress at any point in a plate is

$$\{\sigma\} = \{N\}\ 1/t - \{M\}\ z/D \tag{13}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \qquad \{N\} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}, \qquad \{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

where $\{N\}$ and $\{M\}$ are the forces, and $z$ is the fiber distance at this point. Defining the stress recovery parameters $S$ as

$$S_1 = \frac{1}{t}, \qquad S_2 = \frac{-z}{D} \tag{14}$$

Eq. (13) can be rewritten as

$$\{\sigma\} = \{N\}S_1 + \{M\}S_2 \tag{15}$$

The reciprocals of the section properties of a plate are

$$Z_1 = \frac{1}{t}, \qquad Z_2 = \frac{1}{D} \tag{16}$$

Note that the stress in an element is a linear function of the stress recovery parameters for the element and the forces in the element, which are functions of the reciprocals of the section properties of all of the elements. This can be stated as

$$\{\sigma\} = f[S, N(Z, V), M(Z, V)] \tag{17}$$

The approximation of the stress in an element is the expansion of the stresses in a first-order Taylor series with respect to the intermediate variables $X$ which are the combination of the stress recovery parameters for the element $S$ and the reciprocals of the section properties of all the elements $Z$

$$\{\tilde{\sigma}\} = \{\sigma_0\} + \sum_{i=1}^{\mathrm{NIV}} \frac{\partial\{\sigma\}}{\partial X_i}(X_i - X_{i0})$$
$$\{\tilde{\sigma}\} = \{\sigma_0\} + \sum_{i=1}^{\mathrm{NE}} \sum_{j=1}^{2} \frac{\partial\{\sigma\}}{\partial Z_{ij}}(Z_{ij} - Z_{ij0}) + \sum_{k=1}^{2} \frac{\partial\{\sigma\}}{\partial S_k}(S_k - S_{k0}) \tag{18}$$

where NE is the number of elements. The reciprocals of the section properties were chosen as intermediate variables, rather than the direct section properties, because the element forces are usually close to being linear functions of these quantities. In shape optimization the shape design variables $V$ are also included in the intermediate variables and the term

$$\sum_{i=1}^{\mathrm{NSDV}} \frac{\partial\{\sigma\}}{\partial V_i}(V_i - V_{i0}) \tag{19}$$

where NSDV is the number of shape design variables, must be added to Eq. (18).

The derivatives of the stresses with respect to the intermediate variables are

$$\frac{\partial\{\sigma\}}{\partial Z_{ij}} = S_{10}\frac{\partial\{N\}}{\partial Z_{ij}} + S_{20}\frac{\partial\{M\}}{\partial Z_{ij}}$$
$$\frac{\partial\{\sigma\}}{\partial V_i} = S_{10}\frac{\partial\{N\}}{\partial V_i} + S_2\frac{\partial\{M\}}{\partial V_i} \tag{20}$$
$$\frac{\partial\{\sigma\}}{\partial S_1} = \{N_0\} \qquad \frac{\partial\{\sigma\}}{\partial S_2} = \{M_0\}$$

Note that this approximation is exact for statically determinate structures and captures the cross coupling between changes in the element forces, due to load redistribution, and changes in the element stress recovery parameters. It is of the same order of accuracy of the approximation presented in Eq. (7). The advantage over the approximation presented in Eq. (7) is that only one vector of element stress derivatives needs to be calculated and stored as opposed to two vectors of element force derivatives.

The derivatives of the stress with respect to the intermediate variables are calculated as follows. Rewriting Eq. (15) as

$$\{\sigma\} = [S]\begin{Bmatrix} N(Z, V) \\ M(Z, V) \end{Bmatrix} \tag{21}$$

where

$$[S] = \begin{bmatrix} S_1 & & & S_2 & & \\ & S_1 & & & S_2 & \\ & & S_1 & & & S_2 \end{bmatrix} \tag{22}$$

The element forces are

$$\begin{Bmatrix} N(Z, V) \\ M(Z, V) \end{Bmatrix} = [D(Z)][B(V)]\{u(Z,V)\}^e$$

$$= [D(Z)][B(V)][T(V)]\{u(Z,V)\} \tag{23}$$

where $[D]$ is the force-strain relationship matrix, $[B]$ is the strain-displacement matrix, $\{u\}^e$ are the displacements in the element coordinate system, and $[T]$ is the element coordinate system transformation matrix. Defining $[Q]$ as

$$\{\sigma\} = [S][D][B][T]\{u\} = [Q]^T\{u\} \tag{24}$$

a single stress component can be expressed as

$$\sigma_k = \{Q_k\}^T\{u\} \tag{25}$$

The derivative of this stress component with respect to an intermediate variable is

$$\frac{\partial \sigma_k}{\partial X} = \frac{\partial \{Q_k\}^T}{\partial X}\{u\} + \{Q_k\}^T \frac{\partial \{u\}}{\partial X} \tag{26}$$

From

$$[K(Z, V)]\{u(Z, V)\} = \{P(Z, V)\} \tag{27}$$

where $[K]$ is the global stiffness matrix and $\{P\}$ the global load vector, the derivative of the displacement vector with respect to an intermediate variable is

$$\frac{\partial \{u\}}{\partial X} = [K]^{-1}\left(\frac{\partial \{P\}}{\partial X} - \frac{\partial [K]}{\partial X}\{u\}\right) \tag{28}$$

Substituting Eq. (28) into Eq. (26) gives

$$\frac{\partial \sigma_k}{\partial X} = \frac{\partial \{Q_k\}^T}{\partial X}\{u\} + \{\bar{u}_k\}^T\left(\frac{\partial \{P\}}{\partial X} - \frac{\partial [K]}{\partial X}\{u\}\right) \tag{29}$$

where

$$\{\bar{u}_k\} = [K]^{-1}\{Q_k\} \tag{30}$$

Equation (26) is used to calculate the derivatives of the stress using the direct (state variable) method of sensitivity analysis, or Eq. (29) can be used in the adjoint variable method of sensitivity analysis. Note that if the adjoint variable method is used, only one right-hand side must be solved for each stress. The approximation presented in Ref. 3 requires two right-hand sides, one for each force, for each stress when the adjoint variable method is used.

If the principle, maximum shear, and von Mises stresses need to be calculated, it can be done using the explicit relationships

$$\tilde{\sigma}_{\mathrm{I}} = \frac{\tilde{\sigma}_x + \tilde{\sigma}_y}{2} + \sqrt{\left(\frac{\tilde{\sigma}_x - \tilde{\sigma}_y}{2}\right)^2 + \tilde{\tau}_{xy}^2}$$

$$\tilde{\sigma}_{\mathrm{II}} = \frac{\tilde{\sigma}_x + \tilde{\sigma}_y}{2} - \sqrt{\left(\frac{\tilde{\sigma}_x - \tilde{\sigma}_y}{2}\right)^2 + \tilde{\tau}_{xy}^2}$$

$$\tilde{\tau}_{\max} = \sqrt{\left(\frac{\tilde{\sigma}_x - \tilde{\sigma}_y}{2}\right)^2 + \tilde{\tau}_{xy}^2} \tag{31}$$

$$\tilde{\sigma}_{\mathrm{VM}} = \sqrt{\tilde{\sigma}_x^2 + \tilde{\sigma}_y^2 - \tilde{\sigma}_x\tilde{\sigma}_y + 3\tilde{\tau}_{xy}^2}$$

Note that when a principle stress is calculated three stresses must be approximated. If the method presented in Ref. 3 is used, then six element forces must be approximated. If the principle stresses in both the top and bottom surfaces of an element must be calculated, then six stresses must be approximated. In this case the number of approximated quantities is the same as for the method presented in Ref. 3.

## Optimization Problem Statement

The optimization examples presented in this work are stated as

$$\text{minimize } W[X(Y)] \tag{32}$$

subject to

$$\sigma_{\mathrm{VM}}[X(Y)] \le \sigma_a$$

and

$$Y_i^L \le Y_i \le Y_i^U \qquad i = 1, 2, 3, \ldots, \mathrm{NDV}$$

where $W$ is either the mass or volume of the structure, $\sigma_a$ the allowable stress, $X$ the intermediate variables, $Y$ the design variables, $Y^L$ and $Y^U$ the lower and upper bounds on the design variables, respectively, and NDV is the number of design variables. Note that the intermediate variables are explicit functions of the design variables and that the shape variables are both design variables and intermediate design variables [see Eqs. (18) and (19)].

## Optimization Procedure

The overall optimization procedure used in this paper is as follows.

1) Read in the input data and perform preprocessing.
2) Perform a finite element analysis of the structure.
3) Evaluate the objective function and constraints.
4) Perform a sensitivity analysis of the active and near active stresses with respect to the intermediate variables.
5) Formulate and solve the approximate optimization problem. This will generate the next design.
6) Check for design convergence. If the new design is different from the old design then go to step 2.

The procedure for the solution of the approximate optimization problem is as follows.

1) Calculate the values of the intermediate variables from the design variables.
2) Evaluate the Taylor series approximations for the objective function and constraints.
3) Call the design optimization tools (DOT) optimizer[5] to determine the new values of the design variables.
4) If the approximate optimization problem has converged then go to step 6 of the overall approximate optimization, else go to step 1 of the approximate optimization procedure.

## Examples

Four examples are given here to demonstrate the accuracy of the approximation. All of the examples were solved using the GENESIS[6] structural optimization program.

### Cantilevered Plate

This example consists of finding the minimum mass of the 20-element cantilevered plate shown in Fig. 1. The plate is

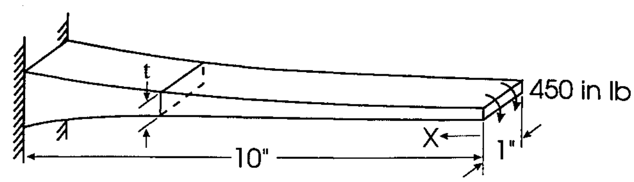


**Fig. 1 Cantilevered plate.**

loaded with a 450.0-in.-lb. moment at the tip and has material properties of $E = 10.0E6$ psi, $\nu = 0.3$, and $\rho = 0.298$ lb/in.$^3$. The 20-element thicknesses are the design variables. The initial element thicknesses are all 1.0 in. There is a displacement constraint on the tip of 0.5 in. and bending stress constraints on the elements of 30,000 psi.

The analytical solution to this problem was obtained in Ref. 7. This problem is statically determinate if a beam model is used and nearly statically determinate with the plate model. Therefore, approximations of the displacement with respect to the reciprocals of the section properties and element forces and the corresponding element stresses with respect to the intermediate variables are nearly exact. The design cycle history for this problem is shown in Table 1. Note that the optimum design is achieved in essentially one design cycle. In Table 2 the final design is compared to the analytic solution found in Ref. 7. Note that the same design was generated in Ref. 3.

### Cantilevered Shell

This example consists of finding the minimum volume of the cantilevered shell shown in Fig. 2. The shell has a uniformly distributed load of a total of 7848 lb applied to the free end, and material properties of $E = 2.9E7$ psi and $\nu = 0.3$. The half-model of structure shown in Fig. 3 was used with symmetric boundary conditions for the optimization. This model has 12 elements along the 90-deg arc and 48 elements along the length of the structure. Initially the shell thickness is 1.0 in. and the radius of each end is 2.5 in. The von Mises stress on the top and bottom of each element is constrained to be less than 10,000 psi. The initial structure has constraints that are violated by 142.2%.

**Table 1 Design cycle history—cantilevered plate**

| | Present method | | Ref. 3 | |
|---|---|---|---|---|
| Design cycle | Objective function | Maximum constraint violation, % | Objective function | Maximum constraint violation, % |
| 0 | 2.980 | 0.0 | 2.980 | 0.0 |
| 1 | 1.076 | 0.0 | 1.076 | 0.0 |
| 2 | 1.074 | 0.0 | 1.074 | 0.0 |

**Table 2 Final design variables—cantilevered plate**

| Variable | Value | Ref. 3 | Ref. 7 |
|---|---|---|---|
| 1 | 0.430 | 0.430 | 0.431 |
| 2 | 0.424 | 0.424 | 0.426 |
| 3 | 0.418 | 0.418 | 0.420 |
| 4 | 0.412 | 0.412 | 0.413 |
| 5 | 0.406 | 0.406 | 0.407 |
| 6 | 0.399 | 0.399 | 0.400 |
| 7 | 0.392 | 0.392 | 0.393 |
| 8 | 0.385 | 0.385 | 0.386 |
| 9 | 0.377 | 0.377 | 0.378 |
| 10 | 0.369 | 0.369 | 0.369 |
| 11 | 0.360 | 0.360 | 0.360 |
| 12 | 0.350 | 0.350 | 0.350 |
| 13 | 0.340 | 0.340 | 0.340 |
| 14 | 0.328 | 0.328 | 0.328 |
| 15 | 0.315 | 0.315 | 0.314 |
| 16 | 0.300 | 0.300 | 0.300 |
| 17 | 0.300 | 0.300 | 0.300 |
| 18 | 0.300 | 0.300 | 0.300 |
| 19 | 0.300 | 0.300 | 0.300 |
| 20 | 0.300 | 0.300 | 0.300 |
| Mass | 1.074 | 1.074 | 1.075 |

**Table 3 Design cycle history—one variable cantilevered shell**

| | Present method | | Ref. 3 | |
|---|---|---|---|---|
| Design cycle | Objective function | Maximum constraint violation, % | Objective function | Maximum constraint violation, % |
| 0 | 78.5 | 142.2 | 78.5 | 142.2 |
| 1 | 90.1 | 40.3 | 92.1 | 39.0 |
| 2 | 100.6 | 7.4 | 101.4 | 7.0 |
| 3 | 103.4 | 0.2 | 104.1 | 0.1 |
| 4 | 103.6 | 0.0 | 104.2 | 0.0 |

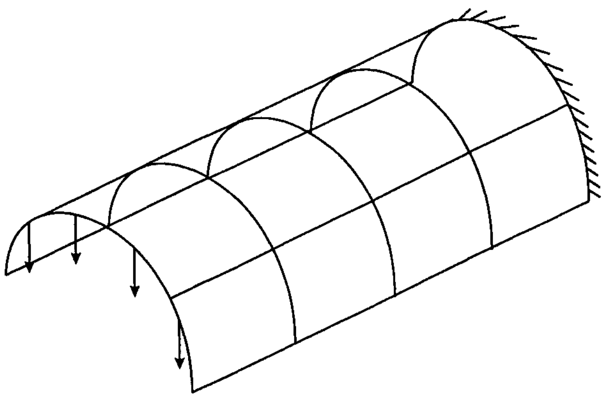

**Fig. 2 Cantilevered shell.**

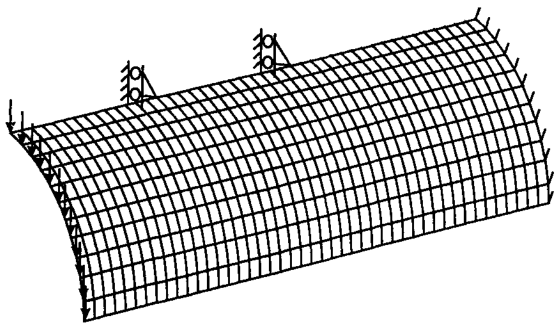

**Fig. 3 Cantilevered shell model.**

Two different optimization problems were solved with this example. In the first the only design variable is the radius of the fixed end. The optimum design is achieved in only four design cycles and has a volume of 103.6 in.$^3$. The optimum radius of the fixed end is 4.10 in. The design cycle history for this problem is shown in Table 3. This problem was solved in Ref. 3 in four design cycles with an optimum radius of 4.06 in. This problem was solved in Ref. 8 using seven finite element analyses with an optimum radius of 4.17 in.

In the second design problem both the fixed and free end radii are design variables as well as the shell thickness. In this case the optimum design is still achieved in only four design cycles and has a volume of 81.5 in.$^3$. The optimum radii of the fixed and free ends are 5.14 and 1.63 in., respectively. The optimum thickness is 0.72 in. In Ref. 3 the problem was solved also in four design cycles and had a volume of 80.9 in.$^3$, radii of the fixed and free ends of 5.15 and 1.61 in., respectively, and thickness of 0.72 in. The design cycle history for this problem is shown in Table 4. A simplified version of this problem was solved in Ref. 8 using 22 finite element analyses. In this version of the problem, the load is proportional to the radius of the free end, causing the design variable for the radius of the free end to go to its lower bound.

In Fig. 4 the approximate von Mises stress in one of the elements is compared to the acutal stress variation in the element from the initial design to the final design of the three design variable model. In this figure 0 represents the initial design and 1 the final design. The approximation is formed at the initial design point. Note that the approximation captures

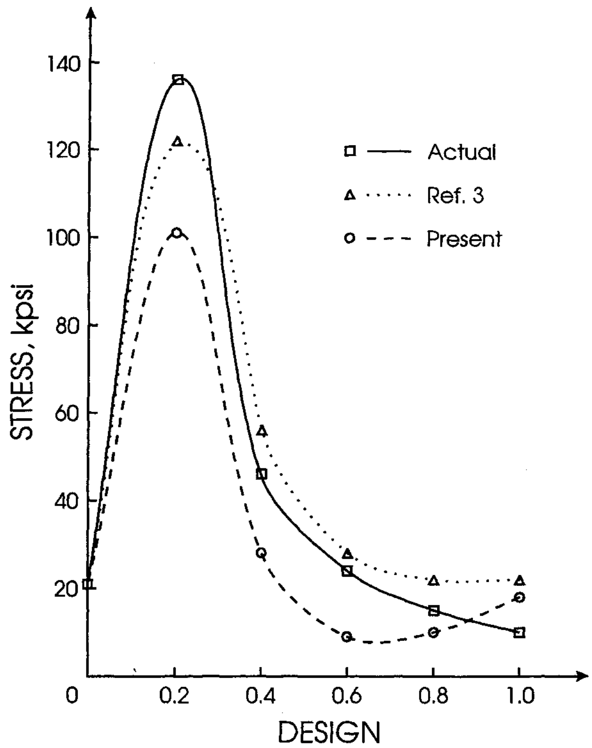


**Fig. 4 Comparison of approximate and actual stress functions.**

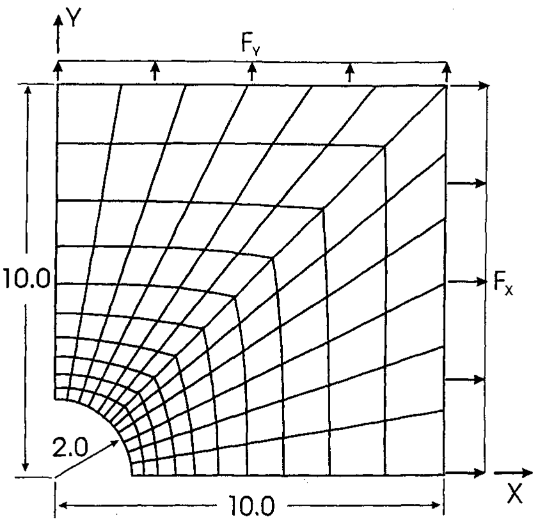


**Fig. 5 Plate with a hole.**

the change in curvature of the actual variation of the stress. Also shown in the figure is the approximate stress from the method presented in Ref. 3. It is shown that both approximations perform well in capturing the nonlinear reponse of the stress to the changes in the three design variables.

#### Plate with a Hole

This example consists of finding the minimum mass of the square plate loaded by unbalanced biaxial tension. A quarter model, shown in Fig. 5, is used due to symmetry. The plate is 20.0 in. on the side and has an initial thickness of 0.15 in. The initial radius of the hole is 2.0 in. The plate is loaded with distributed loads of 900.0 and 450.0 lb/in. on the $X$ and $Y$ edges, respectively, and has material constants of $E = 2.9E7$, $\nu = 0.3$, and $\rho = 0.283$ lb/in.$^3$. The von Mises stress in each element is constrained to be less than 10,000 psi.

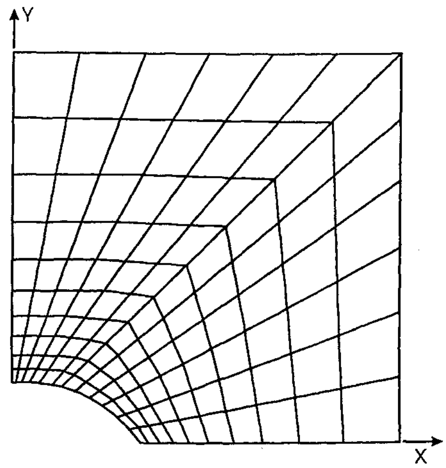


**Fig. 6 Final design of plate with a hole.**

**Table 4 Design cycle history—three variable cantilevered shell**

| | Present method | | Ref. 3 | |
|---|---|---|---|---|
| Design cycle | Objective function | Maximum constraint violation, % | Objective function | Maximum constraint violation, % |
| 0 | 78.5 | 142.2 | 78.5 | 142.2 |
| 1 | 82.3 | 23.6 | 69.2 | 66.8 |
| 2 | 81.7 | 4.1 | 78.8 | 14.1 |
| 3 | 81.9 | 0.2 | 80.4 | 1.9 |
| 4 | 81.5 | 0.0 | 80.9 | 0.0 |

**Table 5 Design cycle history—plate with a hole**

| | Present method | | Ref. 3 | |
|---|---|---|---|---|
| Design cycle | Objective function | Maximum constraint violation, % | Objective function | Maximum constraint violation, % |
| 0 | 4.112 | 39.3 | 4.112 | 39.3 |
| 1 | 4.371 | 0.0 | 4.365 | 0.0 |
| 2 | 3.794 | 0.0 | 3.794 | 0.0 |
| 3 | 3.790 | 0.0 | — | — |

This example has both shape and sizing design variables. The sizing design variable is the thickness of the plate. Four shape design variables are used to control the shape of the hole. The initial design has a mass of 4.11 lb and a maximum constraint violation of 39.3%. The optimum design is found in just three design cycles and has a mass of 3.79 lb. The optimum shape is shown in Fig. 6 and has a thickness of 0.139 in. Essentially, the same design was achieved in two design cycles in Ref. 3. The design cycle histories are shown in Table 5. Simplified plate with hole problems, in which there was no thickness design variable and the plates were loaded with balanced biaxial tension to produce a circular hole, were solved in Ref. 9 and required 6–11 iterations.

## Conclusions

The stress approximation presented in this work is more accurate than a direct approximation of the stress in a linear Taylor series with respect to the design variables or their reciprocals. This is because it captures coupling between changes in the element forces, due to load redistribution, and changes in the element stress recovery parameters. This approximation is exact for statically determinate structures. The accuracy of the approximation is due to the set of intermediate variables used in the Taylor series. The use of this accurate and nonlinear approximation leads to rapid design convergence in both shape and sizing optimization problems. This approximation is of the same order of accuracy of the approximation presented in Ref. 3, but requires the calculation and storage of one-half as much sensitivity information. In the case where principle stresses are approximated on both the top and bottom surfaces of the element, the same amount of sensitivity information is required for both approaches.

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